Basics of Electric Circuits

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1 Alternating Current Circuits

1.1 Using Phasors

There are practical and economic reasons justifying that electrical generators produce *emf with* alternating and sinusoidal form. Therefore, the generators that supply the power grid are AC generators. In these cases voltages, currents and other electrical variables are also sinusoidal waveforms. As an example consider again the RLC serially connected circuit shown in Figure 1 but now the emf is a sinusoidal waveform that is represented by the relation:

$$E(t) = E_{max} cos(\omega t + \alpha) \tag{1}$$

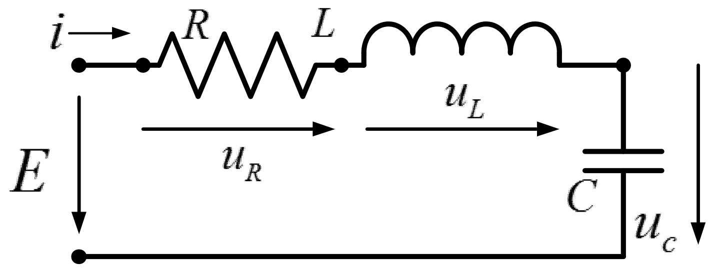

Figure 1 – RLC serial circuit.

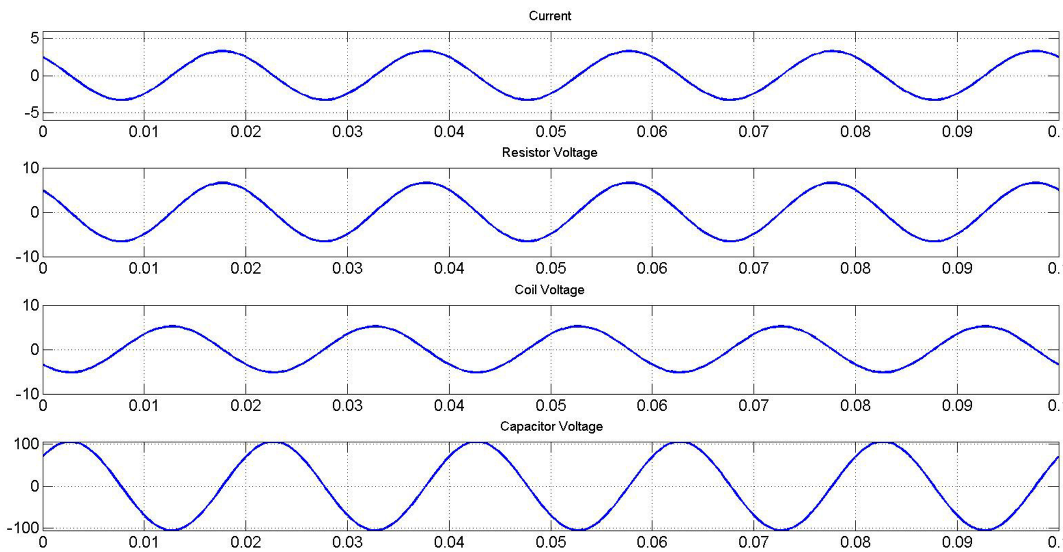

Figure 2 – Evolutions of the current and of the voltages at the resistor, at the coil and at the capacitor terminals.

Solving the circuit, one can verify the evolutions of the current and voltages. In steady state regime – the period after transients have vanished – all the variables are represented by a sinusoidal function (2) similar to the relation (1).

$$x(t) = X_{max} cos(\omega t + \alpha_x) \tag{2}$$

All these waveforms have the same angular frequency ω, but present different amplitudes X_{max} and different phases α_x. In many situations all one needs to know about the solution of a circuit is it's steady state regime, this meaning: the amplitude and the phase of each current or voltage in the circuit. One way to determine the steady state solution is to put a general sinusoidal waveform in the equations that describe the circuit and solve the system of equations in order to determine all the amplitudes and phase angles. This methodology is not friendly to user, as you can verify trying to solve the simple circuit shown in Figure 1.Instead we use an adequate methodology.

The idea is to represent a sinusoidal waveform by a complex number – a phasor. So assume the relation (3) and its representation on Argand plane shown on Figure 3. The instantaneous value, $x(t)$, is equal to the projection of the phasor on real axis. As the angular speed, ω, is the same for all currents and voltages only the phase angles, α_x are important. Accordingly, one can represent the sinusoidal waveform using a complex number as indicated by relation (4).

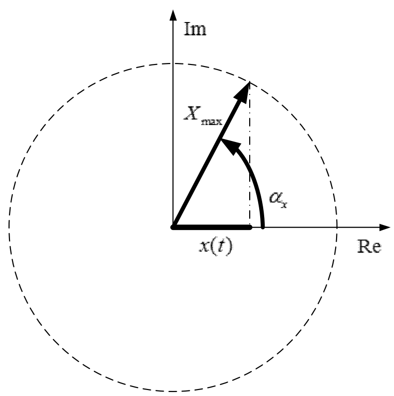

Figure 3 – Phasor representation on Argand plane

$$x(t) = X_{max}\, cos(\omega t + \alpha_x) = Re[(X_{max}e^{j\alpha_x})e^{j\omega t}] \tag{3}$$

$$x(t) = X_{max}\, cos(\omega t + \alpha_x) \quad \Leftrightarrow \quad \bar{X} = X_{max}e^{j\alpha_x}] \tag{4}$$

For practical reasons it is usual to write phasor amplitude by

$$X_{max} = \sqrt{2}X_{RMS} \qquad X_{RMS} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} x^2\, d(\omega t)} \tag{5}$$

where it is used the root medium square value of the sinusoidal waveform.

In sinusoidal steady state regime the ideal elements are represented by the following relations:

Resistor

$$u = Ri \quad \Leftrightarrow \quad \bar{U} = R\bar{I} = \bar{Z}_R\bar{I}$$

Coil

$$u = L\frac{di}{dt} \quad \Leftrightarrow \quad \bar{U} = j\omega L\bar{I} = \bar{Z}_L\bar{I}$$

Capacitor

$$i = C\frac{du}{dt} \quad \Leftrightarrow \quad \bar{U} = -j\frac{1}{\omega C}\bar{I} = \bar{Z}_C\bar{I}$$

The differential equations that describe the behavior of the circuit are transformed in algebraic equations that are easier to solve. The coefficient relating the voltage and the current is a complex number designated by impedance. As an example, the serial circuit shown on Figure 1 1 when in steady state regime is represented by the equation:

$$\bar{E} = \overline{(Z}_R + \bar{Z}_L + \bar{Z}_C)\bar{I} = \bar{Z}_{eq}\bar{I} \qquad \bar{Z}_{eq} = \left[R + j(\omega L - \frac{1}{\omega C})\right]$$

The amplitude of the current depends on the applied voltage and also on the value of the impedance that may depend on the angular frequency. Note also that the current phase shift related to the applied voltage depends on the impedance angle.

1. Problem

Consider the circuit shown on Figure 1 with the following values $R = 2\Omega\, L = 5mH \;\; C = 100\mu F \;\; E_{RMS} = 230V \;\; f = 50H_z$.

a) Find the current and the voltages at the terminals of the resistor, coil and capacitor.
b) Plot the phasors of all these variables on Argand plane.
c) Using the previous results plot the electric power supplied to each element of the circuit and to the circuit itself.

Hints: Note that the electric power is the product of two sinusoidal waveforms, a voltage and a current.

2. Problem

The RLC serial circuit above is supplied with frequencies in the range [20, 80] Hz.

a) Plot the amplitude of the current versus frequency.
b) Plot the phase shift.
c) Plot the module of the equivalent impedance of the circuit.

1.2 Active, Reactive and Apparent Power – Complex Power

The instantaneous power delivered by the supply is calculated by the product of its voltage by the current that it is delivered by it as expressions (6) quantifies. In this expression one introduces the active power and the reactive power defined by expressions (7) and measured, respectively in kW and kVAr units.

$$p(t) = ui = \sqrt{2}U_{RMS}\cos(\omega t+\alpha)\,\sqrt{2}I_{RMS}\cos(\omega t+\alpha-\varphi)$$

$$p(t) = U_{RMS}I_{RMS}\cos(\varphi)\,[1+\cos(2\omega t+2\alpha)] + U_{RMS}I_{RMS}\sin(\varphi)\sin(2\omega t+2\alpha)$$

$$p(t) = P[1+\cos(2\omega t+2\alpha)] + Q\sin(2\omega t+2\alpha)$$

(6)

$$P = U_{RMS}I_{RMS}\cos(\varphi) \qquad Q = U_{RMS}I_{RMS}\sin(\varphi)$$

(7)

It is Also common to use the apparent power as the value to $S = U_{RMS}I_{RMS}$. The value

$$\lambda = \frac{P}{S} = cos(\varphi)$$

(8)

is designed by power factor.

Note that the active power is the average value of the instantaneous power delivered by the supply, that is

$$P = \frac{1}{T}\int_0^T p(t)\,d\tau$$

(9)

The reactive power is a simple way to indicate that the current phasor has a phase shift relative to the voltage.

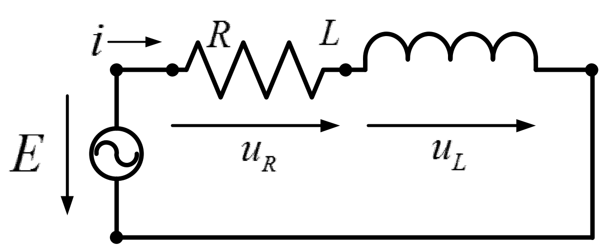

Figure 4 – Inductive circuit.

As an example consider the inductive circuit connected to an AC power supply. The equivalent impedance of this circuit (10) shows that the circuit absorbs active and reactive power.

$$\bar{Z} = Ze^{j\varphi} \qquad z = \sqrt{R^2+(\omega L)^2} \qquad \varphi = atan(\frac{\omega L}{R})$$

(10)

On Argand plane the current phasor is lagged in relation to voltage phasor. In this conditions it is verified that P>0 and Q>0. Figure 5 shows this diagram where it is easy to verify that length *P* is proportional to the active power and length *Q* is proportional to the reactive power.

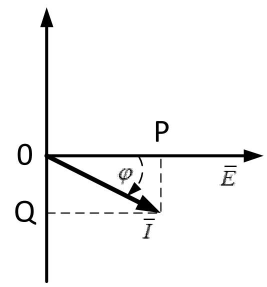

Figure 5 – Phasor diagram of an inductive circuit.

Note that these situations have some drawbacks, as for instance, to supply the same active power, one has a larger current in the feeder when there is a smaller power factor- $\cos\varphi$. This implies that for the same active power supplied, there are larger losses in the feeder than there would be with power factor equal to one. To compensate large phase delay – large consumption of reactive power – one can put a capacitor at the input of the installation. In these cases, one can say that the capacitor supplies reactive power that otherwise would be supplied by the mains. Note also that the current in the capacitor is ahead of the voltage applied to its terminals.

Figure 6 shows the steady state regime of an inductive circuit where the current is time lagging in relation to the voltage. This means that the circuit consumes reactive power, in addition to the active power.

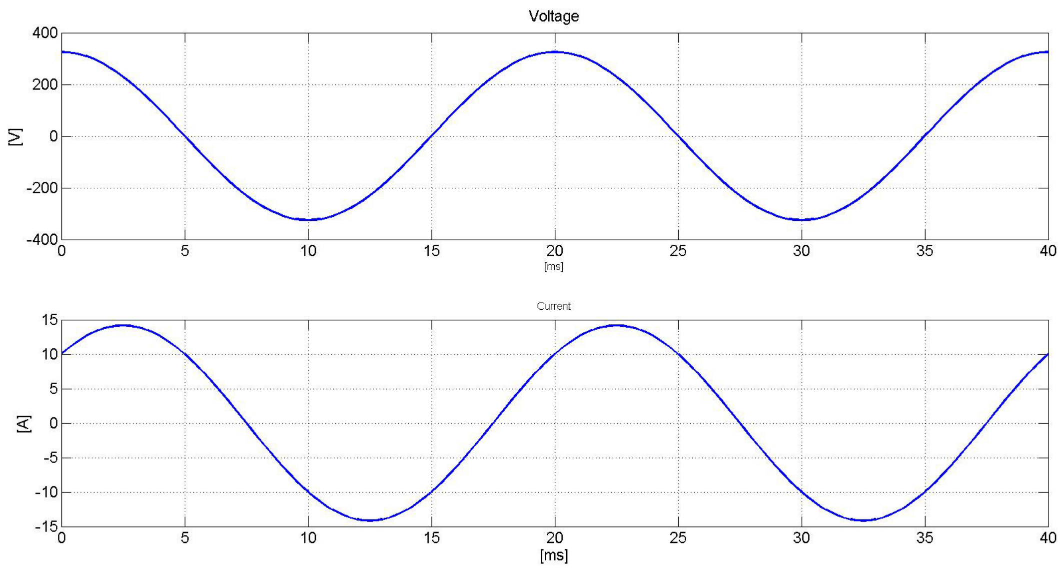

Figure 6 – Steady state regime of an inductive circuit.

Figure 7 presents the time evolution of the instantaneous power delivered to an inductive circuit. The double frequency of this waveform and the negative value of the power, indicating the reverse direction of the power flow, should be underlined. The reverse direction of the power flow in a circuit with only passive elements occurs if there are elements that store energy, like the coils or the capacitors.

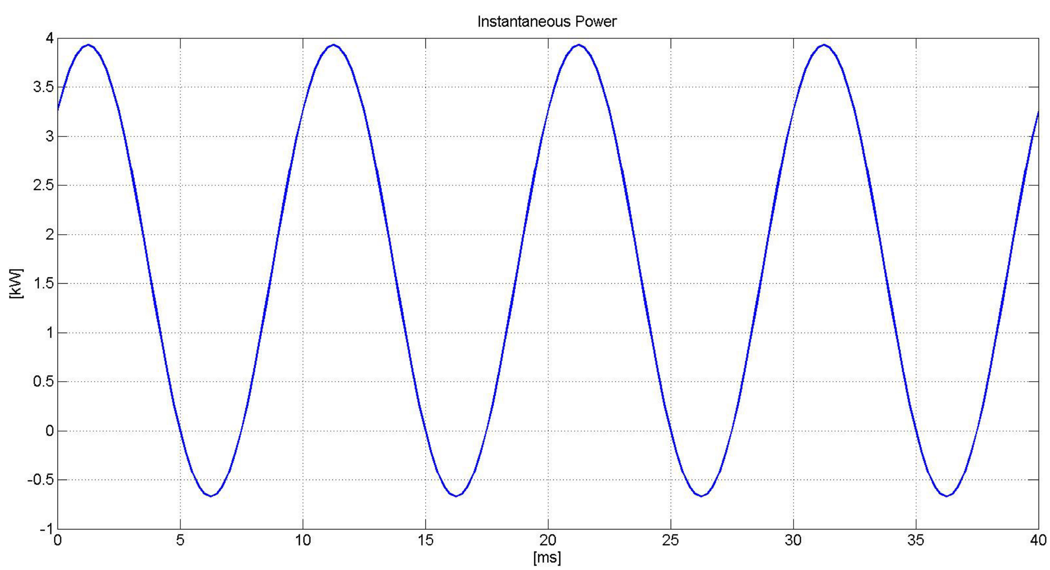

Figure 7 – Evolution of instantaneous power. Note the negative values of the power.

Sometimes it is practical to join in the same variable the active power and the reactive power. This is obtained by introducing the complex power

$$\overline{S} = \frac{1}{2}\overline{E}\,\overline{I}^* = P + jQ = E_{RMS}I_{RMS}\cos(\varphi) + jE_{RMS}I_{RMS}\sin(\varphi) \tag{11}$$

3. Problem

a) Determine the active and reactive power of a coil with an inductance *L* when carry a current *I*.

b) Determine the active and reactive power of a capacitor with a voltage *U* at its terminals.

Hints: Use definition of complex power.

4. Problem

a) Consider an inductive circuit that absorb an active power *P* and a reactive power *Q.* Show that the capacitor with the capacitance equal to

$$C = \frac{Q}{\omega U_{RMS}^2}$$

placed at input terminals totally compensate the $\cos(\varphi)$. That is the voltage and current supplied by source have the same phase.

b) Draw an Argand diagram were are represented the current before and after compensation.

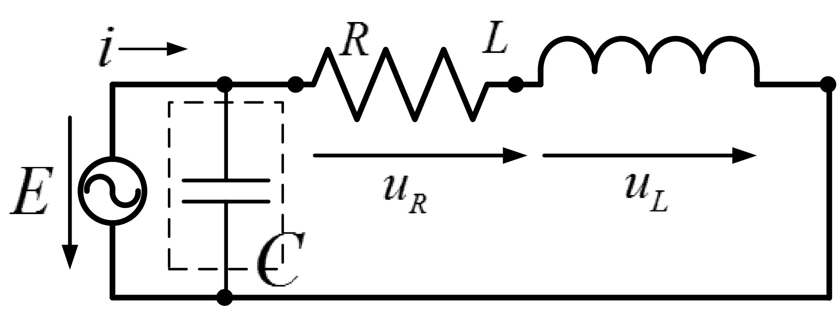

Figure 8 – Capacitor placed at input terminals for $\cos(\varphi)$ compensation.

2 Three Phases Circuits

2.1 Voltages and Currents in a Balanced Circuit

Technical and economic reasons justify the design and the use of three phase systems. This happens in most electric machinery, like generators as well as in the transport and distribution systems.

Figure 9 shows a simple three phase circuit with three power supplies, three conductors – *the phases* - connecting the power supplies to the three phase load. There is another conductor – *the neutral* – connecting the node of the loads to the node of power supplies. In this study it is important to underline a special and usual situation that is the case of a balanced three phase circuit.

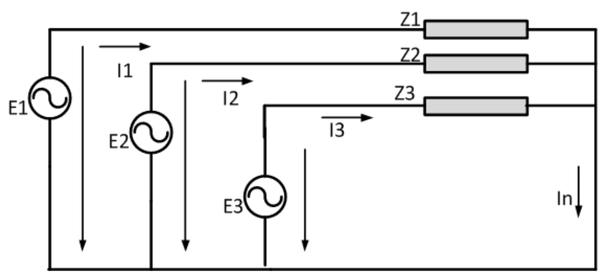

Figure 9 – A three phase system.

In a three phase balanced system, the voltages at power supply have the same amplitude, frequency and its phases are lagged in a regular angle of 120º. In this case one writes the values of the voltages phasors in power supply as

$$\begin{aligned} \bar{E}_1 &= \sqrt{2}E_{RMS}e^{j\alpha} \\ \bar{E}_2 &= \bar{E}_1\, a^2 \qquad a = e^{j2\pi/3} \\ \bar{E}_3 &= \bar{E}_1\, a \end{aligned} \tag{12}$$

The loads are represented by a matrix of impedances[1] and have cycling symmetry that it is common in many electric systems[2], (13).

$$\begin{bmatrix} \bar{E}_1 \\ a^2\bar{E}_1 \\ a\bar{E}_1 \end{bmatrix} = \begin{bmatrix} \bar{Z}_a & \bar{Z}_b & \bar{Z}_c \\ \bar{Z}_c & \bar{Z}_a & \bar{Z}_b \\ \bar{Z}_b & \bar{Z}_c & \bar{Z}_a \end{bmatrix} \begin{bmatrix} \bar{I}_1 \\ \bar{I}_2 \\ \bar{I}_3 \end{bmatrix} \tag{13}$$

[1] The not null elements out of main diagonal are due, for instance, to magnetic coupling.
[2] These equations result from KVL law.

Simple calculation enables us to verify the following results:

$$\overline{I}_1 = \frac{\overline{E}_1}{\overline{Z}_a + a^2\overline{Z}_b + a\overline{Z}_c} \qquad \overline{I}_2 = a^2\overline{I}_1 \qquad \overline{I}_3 = a\overline{I}_1 \qquad \overline{I}_n = 0 \tag{14}$$

1. Problem

Demonstrate the previous result.

Hints: Verify these relations are true $\overline{I}_2 = a^2\overline{I}_1 \quad \overline{I}_3 = a\overline{I}_1$ and apply the KCL law to one node in the circuit.

The three phase system makes available two different values of voltages: the single phase voltage E_j that is a phase to neutral potential difference and the phase to phase voltage that is the phase to phase potential difference

$$\bar{E}_{12} = \bar{E}_1 - \bar{E}_2 \quad \bar{E}_{23} = \bar{E}_2 - \bar{E}_3 \quad \bar{E}_{31} = \bar{E}_3 - \bar{E}_1 \tag{15}$$

2. Problem

Verify that the relations between the RMS of the phase to phase voltage and single phase voltage in a balanced three phase system is equal to

$$E_{\Delta RMS} = \sqrt{3}\, E_{RMS} \tag{16}$$

It is interesting to underline that the currents are also a three phase balanced system as well as the applied voltages. Besides, the current in the neutral conductor becomes equal to zero. In many situations the conductor of the neutral is removed or this conductor has a smaller section than the section of the conductors of the phases.

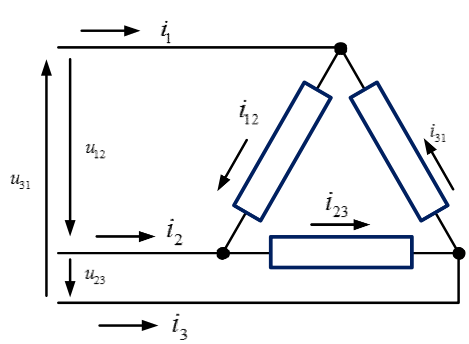

Figure 10 – Delta connection.

In Figure 9, three phase load has a common terminal – the neutral – and this kind of connection is named star connection. Similar situation occurs at power supply. However, the connections may have a different topology. For instance, the Figure 10 presents the three phase load with delta connection.

3. Problem

Consider a balanced three phase system with a delta connected load. Establish the relation between the RMS values of the phase and load currents.

Hints: Consider a cycling symmetry for load impedances matrix and voltages are a balanced three phase system.

2.2 Active, Reactive, Apparent Power and Complex Power

The instantaneous power delivered by the power sources is equal to the sum of the power delivered by all phases. So, one writes the following expressions:

$$\begin{aligned} p &= e_1 i_1 + e_2 i_2 + e_3 i_3 \\ p &= P_1\left[1+\cos(2\omega t + 2\alpha)\right] + Q_1 \sin(2\omega t + 2\alpha) + \\ &\quad P_2\left[1+\cos(2\omega t + 2\alpha - 4\pi/3)\right] + Q_2 \sin(2\omega t + 2\alpha - 4\pi/3) + \\ &\quad P_3\left[1+\cos(2\omega t + 2\alpha - 2\pi/3)\right] + Q_3 \sin(2\omega t + 2\alpha - 2\pi/3) \end{aligned} \tag{17}$$

$$p = P = 3E_{RMS} I_{RMS} \cos(\varphi) \tag{18}$$

It is important to note that the instantaneous power in the three phase balanced system is constant. There are no oscillations; it is therefore a different situation from that of single phase circuits. In the three phase circuits, the instantaneous power is equal to the active power.

4. Problem

Consider an electrical motor as a device that transform electrical energy to the mechanical energy. Discuss the performance of a single phase motor versus three phase balanced motor in what concerns the mechanical energy, to the torque and to the speed.

Hints: Consider a full efficiency energy conversion and remember that mechanical power is given by the expression $p_m = \omega_m T$.

The complex power for a three phase circuit is determined by the expression (19) that assumes the form when the system is balanced.

$$\overline{S} = \frac{1}{2}\overline{E}_1\overline{I}_1^* + \frac{1}{2}\overline{E}_2\overline{I}_2^* + \frac{1}{2}\overline{E}_3\overline{I}_3^* \tag{19}$$

$$\overline{S} = P + jQ = 3E_{RMS}I_{RMS}\cos(\varphi) + j3E_{RMS}I_{RMS}\sin(\varphi) \tag{20}$$

All the phase variables – current and voltage – have the equal waveforms in all phases. Only presents different phase shifts. So, it is usual to represent a three phase circuit by an equivalent single phase circuit as the one shown on Figure 11. Note that electric power in this circuit is only a third part of the total. Besides, the conductor *N* may not exist or may have a different impedance of those of the phase conductors.

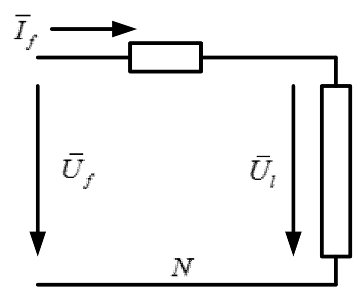

Figure 11 – Equivalent single phase circuit of a balanced three phase circuit.

5. Problem

Consider a balanced three phase system. Show that input single phase impedance may be calculated by

$$\bar{Z}_{eq} = \frac{U_{\Delta}^2}{\bar{S}} \tag{21}$$